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Note

A Note on Variational–Iterative Schemes Applied to Burgers' Equation

This paper proposes a technique for solving non-linear differential equations such as those which govern viscous fluid flow. The aim is to isolate a linear self-adjoint operator, e.g., the Laplacian, from the rest of the equation and then to construct a variationally equivalent formulation which can be solved iteratively. The test example chosen is the steady-state version of Burgers' equation. The results are discussed and measures of convergence of the method are obtained. © 1985 Academic Press, Inc

1. Introduction

Complementary variational principles have been introduced by Kato [8] and others up to Walpole [9] for the equation

$$T\phi = f \tag{1}$$

in a suitably chosen real Hilbert space H where T is a completely continuous self-adjoint operator and f is some function belonging to H. Further work done by Burrows and Perks [5, 6] is an application of these earlier ideas. These principles provide upper and lower bounds for $\langle \phi \mid f \rangle$ where $\langle\ \rangle$ denotes the inner product of H and the theory has been applied to quantum mechanical scattering problems.

Arthurs and Robinson [4], Arthurs [1], Arthurs and Anderson [2] and Arthurs and Coles [3] have also introduced complementary variational principles for the solution of equations of the form

$$T\phi = f(\phi) \tag{2}$$

where f may be a non-linear function of ϕ. These provided upper and lower bounds to a functional when certain conditions are satisfied. Under suitable boundary conditions, complementary extremum principles can be found when

$$T - \frac{df}{d\phi} > 0 \tag{3}$$

or

$$T - \frac{df}{d\phi} < 0. \tag{4}$$

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More can be said about (3) and (4) when the operator T is known to be positive or negative, e.g., if T is positive then the condition $df/d\phi < 0$ is sufficient for (3) to hold, but it is not necessary.

In later work Burrows and Perks [7] have extended their linear theory to deal with the non-linear equation (2). A variational-iterative scheme is used to deal with the problems provided by Eq. (2) so that the simpler linear theory is applied to a sequence of equations, the solutions of which converge to the solutions of Eq. (2).

By considering the functional

$$J(\Delta, \Phi) = \langle \Phi \mid T\Phi \rangle - 2\langle \Phi \mid f \rangle + \Delta \langle T\Phi - f \mid T\Phi - f \rangle \tag{5}$$

where Δ is a real constant, Burrows and Perks demonstrate that the real quantity

$$S = \min_{\psi \in H} \{J(\Delta_1, \omega_p) - J(\Delta_2, \psi)\} \tag{6}$$

provides a measure of the convergence criteria. Here ψ denotes the exact solution, Δ_1, Δ_2 refer to the minimum and maximum principles, respectively, and ω_p denotes the limit of the sequence of trial functions $\{\omega_{n,p}\}$ containing p variational parametes for each iterate. In some cases bounds for $\langle \phi \mid f(\phi) \rangle$ are required where $T\phi = f(\phi)$ and the calculations also provide approximate bounds for the quantity.

In applying this work to the solution of non-linear equations, the basic idea is to attempt to rearrange the non-linear equation

$$A\phi = f(\phi) \tag{7}$$

into the form (2) where T is self-adjoint on the space considered and such that T has a discrete spectrum. Then we iterate with the sequence of equations

$$T\psi_{n+1} = f(\Phi_{n+1}) \tag{8}$$

obtaining Φ_{n+1} as a variational approximation to ψ_{n+1}. Under certain conditions the sequence $\{\Phi_{n+1}\}$ will converge to ϕ. A discussion of acceleration of convergence and choice of Φ_0 to start the procedure is given by Burrows and Perks [5]. To produce convergence Eq. (1) is often rewritten as

$$T\phi = bT\phi + (1 - b)\, T\phi \tag{9}$$

and Eq. (8) now becomes

$$T\psi_{n+1} = T\phi_n + b\{f(\phi_n) - T\phi_n\} \tag{10}$$

where b is a constant chosen to produce rapid convergence.

In this paper we demonstrate how these principles can be used by applying them to the steady-state case of Burgers' equation which is an important equation in fluid dynamics. In fact, Burgers' equation has been extensively used in the past as a test example for much numerical work.

2. Application to Burgers' Equations

Because of its similarity to the Navier–Stokes equation Burgers' equation, namely,

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=v\frac{\partial^2 u}{\partial x^2} \tag{11}$$

where $u=u(x,t)$ in some domain and v is a parameter, often arises in the mathematical modelling used to solve problems in fluid dynamics involving turbulence. The limitations of the analytical solution for certain values of the parameter v due to slow convergence means that numerical approaches become necessary. In order to simplify matters we show how the variational principles already mentioned can be applied to the steady-state version of Burgers' equation, namely,

$$u\frac{du}{dx}=v\frac{d^2u}{dx^2}, \qquad x\geqslant 0 \tag{12}$$

under the boundary conditions

$$u(0)=0, \qquad u(\infty)=-2v. \tag{13}$$

(Exact solution is $u=-2v\tanh x$.)

The substitution $y=1-e^{-x}$ transforms the infinite domain $[0,\infty]$ to the finite domain $[0,1]$ and the boundary conditions become

$$\phi(0)=0, \qquad \phi(1)=-2 \qquad (0\leqslant y\leqslant 1) \tag{14}$$

on making the substitution $\phi=u/v$.

Equation (12) then becomes

$$-\frac{d^2\phi}{dy^2}=-y\frac{d^2\phi}{dy^2}-(1+\phi)\frac{d\phi}{dy} \tag{15}$$

with exact solution

$$u=\frac{2\{(1-y)^2-1\}}{(1-y)^2+1}. \tag{16}$$

This solution also satisfies the condition $\phi^1(1)=0$.

We now attempt to find the variational solution of Eq. (15) over the domain $0\leqslant y\leqslant 1$ and compare results with the exact solution.

Consider the Hilbert space of functions which satisfy the boundary conditions $\phi(0)=0$, $\phi^1(1)=0$ and let the inner product be defined by

$$\langle\phi_1\mid\phi_2\rangle=\int_0^1\phi_1(y)\,\phi_2(y)\,dy.$$

Then the operator $T = -d^2/dy^2$ is symmetric since

$$\langle \phi_2 \mid T\phi_1 \rangle = \langle T\phi_2 \mid \phi_1 \rangle.$$

The eigenvalues of T are defined by

$$\frac{-d^2\phi_i}{dy^2} = \lambda_i \phi_i \tag{17}$$

which, on imposing the boundary conditions, leads to

$$\lambda_i = \left[\frac{(2i+1)\pi}{2}\right]^2 \qquad (i = 0, 1, 2, \ldots) \tag{18}$$

and

$$\phi_i = a_i \sin \frac{(2i+1)\pi y}{2}. \tag{19}$$

This would suggest that we take $\phi_0(y) = a_0 \sin(\pi y/2)$ as a first approximation in the variational approach.

We rewrite (15) in the form (2) where

$$f(\phi) = -y \frac{d^2\phi}{dy^2} - (1+\phi)\frac{d\phi}{dy}. \tag{20}$$

The eigenvalues λ_i of T are discrete and $\lambda_i \geqslant (\pi/2)^2$ so we will take $\Delta = \Delta_1 = 0$ and $\Delta = \Delta_2 = -(2/\pi)^2$ in $J(\Delta, \phi_{n+1})$ to obtain minimum and maximum principles, respectively, at the unique stationary point $\phi_{n+1} = \psi_{n+1}$. We use

$$J(\Delta_2, \psi) \leqslant -\langle \phi \mid f \rangle \leqslant J(\Delta_1, \phi) \tag{21}$$

where $J(\Delta_1, \Phi)$ represents a minimum principle at the unique stationary point $\Phi = \phi$ with

$$\Delta = \Delta_1 \geqslant -\frac{1}{\lambda_i} \qquad \text{for all } i$$

and $J(\Delta_2, \psi)$ represents a maximum principle at the unique stationary point $\psi = \phi$ with

$$\Delta = \Delta_2 \leqslant -\frac{1}{\lambda_i} \qquad \text{for all } i.$$

This gives

$$J(\Delta_2, \psi) \leqslant -\langle \phi \mid f \rangle \leqslant J(\Delta_1, \Phi) \tag{22}$$

and in our particular case

$$J\left(-\frac{4}{\pi^2}, \Phi_{n+1}\right) \leqslant -\langle \psi_{n+1} \mid f(\phi_n) \rangle \leqslant J(0, \Phi_{n+1}). \tag{23}$$

Defining the functional by

$$G(\Phi_{n+1}) \equiv J(0, \Phi_{n+1}) \tag{24}$$

we first apply this variational-iterative approach using the one-parameter trial function

$$\omega_{n,1} = a_{n,1} \sin \frac{\pi y}{2}$$

to Eq. (15) subject to the boundary conditions (14).

3. Discussion of Results

The results obtained for the one-parameter trial function suggest that we require more complex trial functions and the calculations were repeated with trial functions of the form

$$\omega_{n,m} = \sum_{j=1}^{m} a_{n,m}^{(j)} \sin \frac{(2j-1)}{2} \pi y \tag{25}$$

where the form of $\omega_{n,m}$ is suggested by the eigenfunctions in Eq. (19).

In each case the melasure of convergence is provided by

(a) $\{G(\phi) - G(\phi_p)\}$,

(b) $S_p = -\frac{4}{\pi^2} \langle T\omega_p - f(\omega_p) \mid T\omega_p - f(\omega_p) \rangle$ (26)

where ϕ_p is the best p-parameter variational approximation obtained.

It is important to explain the algorithm used to find the p-parameter variational approximation ϕ_p as the procedure used, although simple, has not been used before in this type of work. Starting from the $(p-1)$-parameter trial function given by Eq. (25) with $m = p - 1$ the steps is the procedure are as follows:

(i) Let $i = p - 1$; i is a counter;

(ii) Form the p-parameter trial function

$$\Phi_{n,p} = \omega_{n,p} + a_{n,p} \sin \frac{(2i+1)\pi y}{2}$$

where $a_{n,p}$ is to be determined in the next step;

(iii) Use the normal p-parameter variational approximation

$$\Phi_p = \lim_{n \to \infty} \Phi_{n,p}$$

and calculate the corresponding measure of convergence S_p, say;

(iv) If $S_p \geqslant S_{p-1}$ increase i by 1 and return to step (ii); S_{p-1} is the measure of convergence given by $\omega_{p-1} = \lim_{n \to \infty} \omega_{n,p-1}$;

(v) When $S_p < S_{p-1}$ stop since we have found the final p-parameter variational approximation Φ_p.

This approach yields approximate variational solutions ϕ_p containing p-parameters whose fit to the exact solution increases with p. The approach is inductive. To find ϕ_p we use the approach based on the functional

$$G(\omega_{n,p}) = \langle \omega_{n,p} \mid T\omega_{n,p} \rangle - 2\langle \omega_{n,p} \mid \bar{f}(\omega_{n-1,p}) \rangle \tag{27}$$

where

$$\bar{f}(\omega_{n-1,p}) = (1-b)\, T\omega_{n-1,p} + bf(\omega_{n-1,p}) \tag{28}$$

and proceed in the usual way with a suitable choice of the initial values for the iterative scheme. This ensures rapid convergence of the variational scheme.

This method has been used to generate variational approximations Φ_p of increasing accuracy, as measured by S_p, for $p = 3$ up to $p = 7$. Table I shows the variations of the measures of convergence S_p and $\{G(\phi) - G(\phi_p)\}$ with p. Clearly the variational method is converging. This is confirmed by Table II which shows the variation of the error in the p-parameter approximation $\{\phi(y) - \phi_p(y)\}$ with $y = 0(0.1)1$.

The encouraging results obtained for the steady-state case of Burgers' equation gives confidence in the possible application of complementary variational principles to Burgers' equation itself and this work is in progress.

TABLE I

Variation of the Measures of Convergence $\{G(\phi) - G(\phi_p)\}$ and S_p with p, where S_p Is Given by Eq. (26), and ϕ_p Is the Best p-Parameter Variational Approximation of the Text

p	$\{G(\phi) - G(\phi_p)\}$	S_p
1	+3.8681	−1.1296
2	−0.1185	−0.1059
3	−0.0416	−0.0725
4	−0.0160	−0.0576
5	−0.0045	−0.0488
6	−0.0016	−0.0429
7	−0.0004	−0.0394

TABLE II

The Variation of the Error $\varepsilon_p(y) = \{\phi(y) - \phi_p(y)\} \times 10^4$ of ϕ_p, the Best p-Parameter Variational Approximation Discussed in the Text

y	ε_1	ε_2	ε_3	ε_4	ε_7
0.1	+1969	+118	+25	+27	+34
0.2	+3646	−67	−9	−20	−36
0.3	+4960	−174	−53	−20	−5
0.4	+5874	−250	−115	−84	−58
0.5	+6389	−281	−164	−113	−87
0.6	+6556	−276	−149	−98	−69
0.7	+6474	−257	−86	−43	−9
0.8	+6271	−242	−52	+8	+6
0.9	+6081	−234	−74	−15	−12
1.0	+6006	−232	−95	−50	−23

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J. Caldwell

*School of Mathematics,
Statistics and Computing
Newcastle upon Tyne Polytechnic
Newcastle upon Tyne, England*

R. Saunders
P. Wanless

*Department of Mathematics and
Computer Studies, Sunderland
Polytechnic, Sunderland, England*